Given $\dot{x} = f(x)$; $x(0) = x_0$. Find the solution $x(t)$.

1 Analytically : (rare)

example: $\dot{x} = ax$; $x(0) = x_0$ $a, x_0 \in \mathbb{R}$

technique : separation of variables

$$\frac{dx}{dt} = ax \Rightarrow \int \frac{dx}{x} = \int a\, dt + c$$

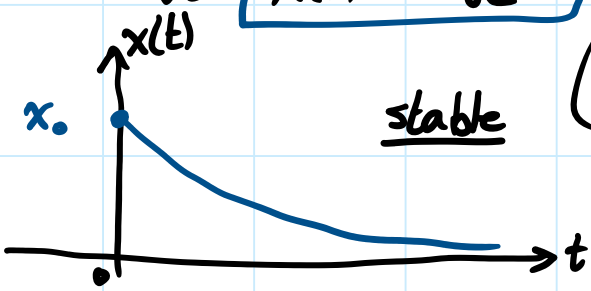
$$\Rightarrow e^{\ln|x|} = e^{(at + c)}$$

$$\Rightarrow |x(t)| = e^{at} \underbrace{e^{c}}_{k} \Rightarrow x(t) = k e^{at}$$

$$x(0) = \boxed{k = x_0} \quad \therefore \quad \boxed{x(t) = x_0 e^{at}}$$

if $a < 0$

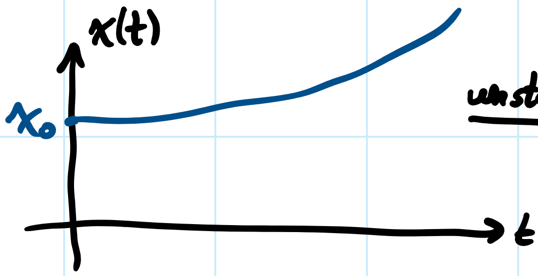
$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x_0 e^{at} = 0$$



stable ($\bar{x} = 0$ Fixed pt)

if $a > 0$

$$\lim_{t \to \infty} x(t) = \infty$$



unstable

special case: if $x_0 = 0$

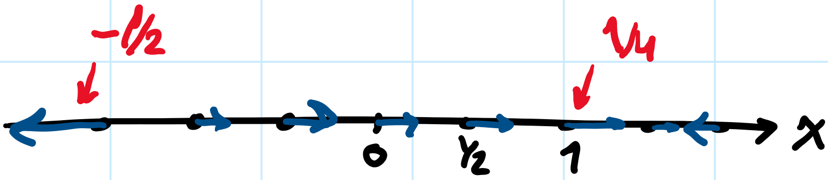
$x(t) = x_0 e^{at} = 0$ Fixed point but unstable!

② **Numerically:** If the D.E. is not analytically solvable

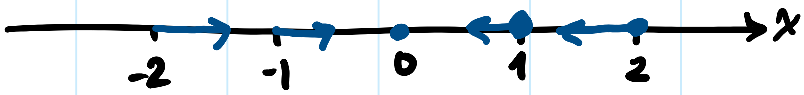
Euler's Method (the intuitive approach)

Vector Field: It is a field of vectors that completely defines a function.

example in 1D:



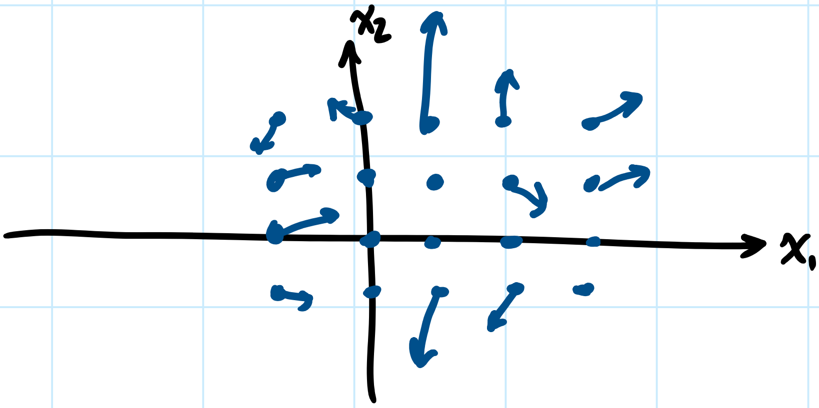
$f: \mathbb{R} \to \mathbb{R}$: $f(x) = -2x$



example in 2D:

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f(x_1, x_2) = \begin{bmatrix} x_1^2 + x_2^3 \\ x_2 \end{bmatrix}$$

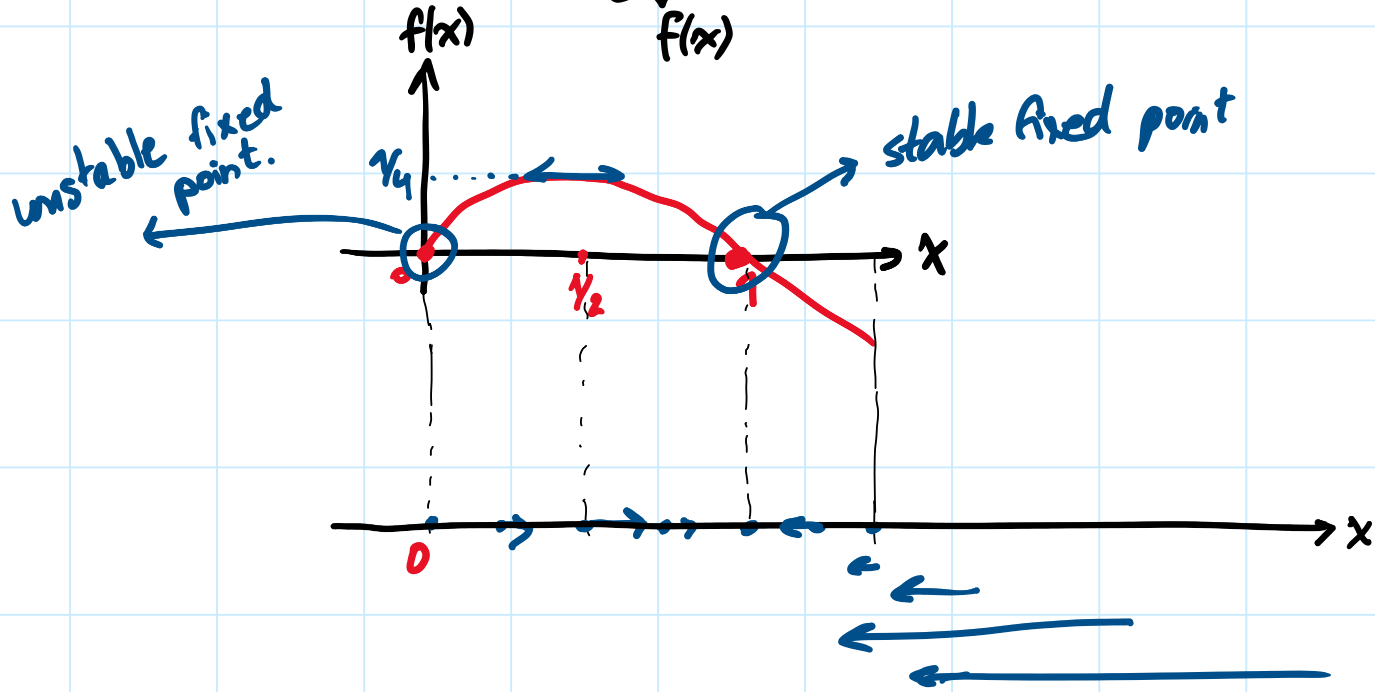


Question: How is all of this related to solving D.E.?

Example modeling population growth

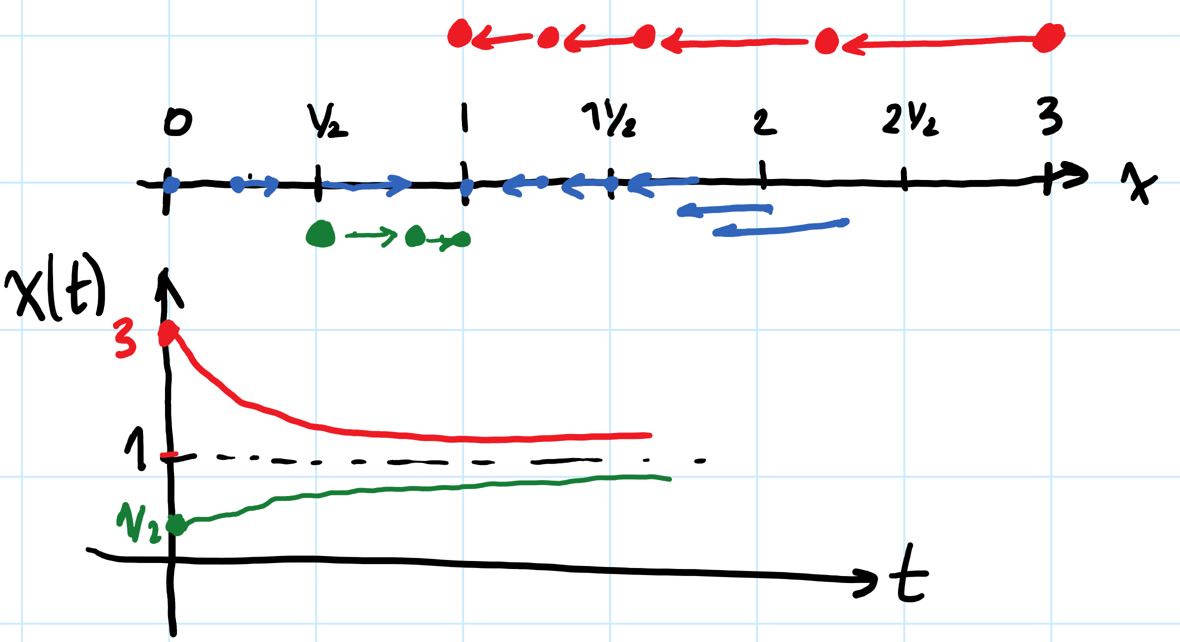
$\dot{x} = x \qquad \frac{\dot{x}}{x} = 1-x \qquad x \geq 0 \qquad x \in \mathbb{R}^+$

$\Rightarrow \dot{x} = \underbrace{x(1-x)}_{f(x)}$ .... Logistic Model

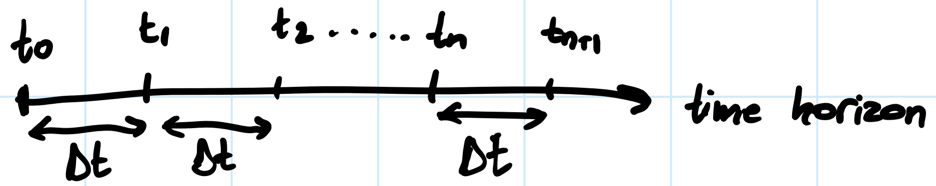


solve graphically $\dot{x} = x(1-x)$

$x(0) = 1/2$



Goal: Build a mathematical framework for the intuition.



$x_n \approx x(t_n)$ $\qquad$ $x(t_n)$: exact solution, x_n : approx.

Algorithm: Given x_n; how to find x_{n+1}?

$$x_{n+1} = x_n + \underbrace{\text{push of vector } (@\, x_n)}_{= \Delta t \cdot f(x_n)}$$

$\therefore$ **Euler's Method:** $\quad x_{n+1} = x_n + \Delta t\, f(x_n)$

$$\dot{x} = f(x); \quad x(0) = x_0$$

Step by step:

(1) $\quad x_1 = x_0 + \Delta t\, f(x_0)$

(2) $\quad x_2 = x_1 + \Delta t\, f(x_1)$

$\vdots$

(n) $\quad x_n = x_{n-1} + \Delta t\, f(x_{n-1})$

(n+1) $\quad x_{n+1} = x_n + \Delta t\, f(x_n)$

$\vdots$

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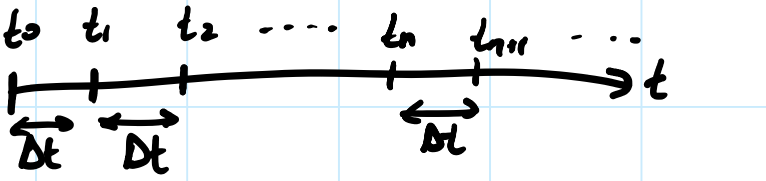
Euler's Method (the mathematical approach)

$\dot{x} = f(x) \; ; \; x(0) = x_0$



$t_n = t_0 + n\Delta t$

key ingredient for Euler's Method

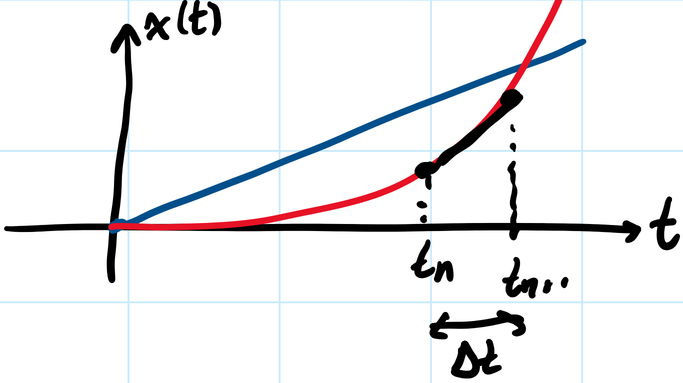
$$\dot{x}(t_n) = \cancel{\lim_{\Delta t \to 0}} \frac{x(t_n + \Delta t) - x(t_n)}{\Delta t}$$

$x(t) = t^2$

$\dot{x} = 2t$

$x(t) = t$

$\dot{x}(t) = 1$



$\dot{x}(t) = f(x)$

$\wr\wr$

@ t_n $\quad \dfrac{x(t_n + \Delta t) - x(t_n)}{\Delta t} = f(x(t_n))$

$$\frac{x_{n+1} - x_n}{\Delta t} = f(x_n) \longrightarrow \boxed{x_{n+1} = x_n + \Delta t f(x_n)}$$

push

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Modified Euler Method:

trial step $\tilde{x}_{n+1} = x_n + \Delta t f(x_n)$

real step: $x_{n+1} = x_n + \frac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1})\right]\Delta t$

Runge-Kutta Method:

$$x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where
$$\begin{cases} k_1 = f(x_n)\,\Delta t \\ k_2 = f(x_n + \frac{1}{2}k_1)\,\Delta t \\ k_3 = f(x_n + \frac{1}{2}k_2)\,\Delta t \\ k_4 = f(x_n + k_3)\,\Delta t \end{cases}$$

Error of approximation

$$E_n := |x(t_n) - x_n|$$

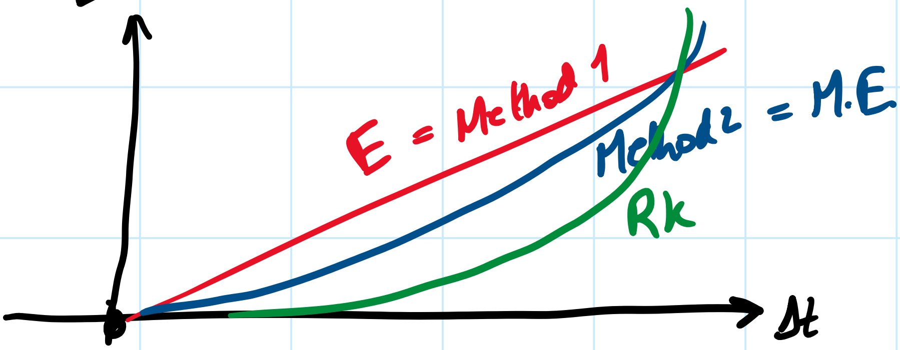
$x(t_n)$: exact solution

x_n: approximated solution for the numerical method

$E_n \propto \Delta t$ Euler's method

$E_n \propto \Delta t^2$ Modified Euler

$E_n \propto \Delta t^4$ Runge-Kutta (4th order)



$$E_n \xrightarrow[\Delta t \to 0]{} 0$$